

Nonstandard general relativity

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Abstract

We show that Einstein's equations in a nonstandard physical gauge have vacuum solutions with an asymptotically flat rotation curve as it is observed in the dark halos of galaxies. Introducing a material disk into this model we find a matter density in accordance with the Tully-Fisher relation.

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1 Introduction

General relativity is a classical gauge theory. This implies that the fundamental fields, like the metric $g_{\alpha\beta}$, are not directly observable. Therefore, in investigating gravitational effects it is important to identify the observable quantities which are actually measurable. Every observable is defined by a measuring process. On the scale of galaxies one most important observable is the circular velocity $V(r)$ of stars or gas which can be measured by the Doppler shift of spectral lines (r is the radius of the circular orbit).

In the theoretical analysis one should try to relate the observables to the fundamental fields, say to $g_{\alpha\beta}(x)$. The best would be if the metric under certain assumptions can be uniquely expressed by the observables. Then the gauge ambiguity has been removed, the gauge is fixed by a physical requirement. The other possibility is to choose the gauge on unphysical grounds, for example by some geometric convention or/and to simplify the solution of the differential equations. This is dangerous because one might miss some important physics. Our program of fixing the gauge by observables is a sort of inverse procedure compared with standard general relativity where one first calculates a metric by solving Einstein's equations in some special gauge and then determines the observables. Clearly, in standard general relativity one cannot be sure that one finds all physically relevant solutions. Indeed we are going to show that Einstein's equations have *vacuum* solutions with an asymptotically flat rotation curve $V(r)$. These nonstandard solutions can be used to describe the dark halo of galaxies without introducing hypothetical dark matter. After the recent Xenon-experiment has again not found any signal of dark matter particles [1] one should seriously investigate nonstandard general relativity.

The paper is organized as follows. As preliminaries we first consider the rotation curve in a general spherically symmetric metric. Although this might be widely known we do not know a good reference. In Sect.3 we solve the vacuum Einstein's equations in our general spherically symmetric setting and express the metric tensor by the circular velocity $V(r)$. From these vacuum solutions we construct in Sect.4 solutions with a disk of ordinary matter by means of the well-known displace, cut, and reflect method. This gives a simple model of a spiral galaxy. Assuming a circular velocity which is constant $= V_{\text{flat}}$ for large r , we find a matter density proportional to V_{flat}^4 . This is in accordance with the Tully-Fisher relation [2] [3].

2 The circular velocity in general relativity

We consider a star moving in an arbitrary static gravitational field. Following Weinberg ([4], chapter 3/2) we introduce the freely falling coordinate system ξ^α of the moving star. In this system the star is at rest and the observer on earth moves with the 4-velocity

$$u^\alpha = \frac{d\xi^\alpha}{d\tau} = (u^0, \vec{u}), \quad u^0 = (1 - \vec{V}^2)^{-1/2}, \quad \vec{u} = u^0 \vec{V}, \quad (2.1)$$

because in the locally inertial coordinates special relativity holds. Here τ is the proper time

$$d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta, \quad (2.2)$$

$\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$ is the Minkowski tensor and the speed of light is put $= 1$. In addition Weinberg introduces a laboratory coordinate system x^μ which in our case is attached to the observers telescope. To simplify the following discussion we assume that the astronomer on earth has corrected his measurements for the motion of the earth with respect to the center of the galaxy, so that we can choose the center of the galaxy as origin of the laboratory coordinate system. Now the star moves on a geodesic ([4], eq.(3.2.3))

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \quad (2.3)$$

where $\Gamma_{\alpha\beta}^\mu$ are the Christoffel symbols of the metric $g_{\mu\nu}$. The latter is defined by ([4], eq.(3.2.7))

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \eta_{\alpha\beta} \frac{\partial \xi^\beta}{\partial x^\nu}. \quad (2.4)$$

Since the local inertial coordinates ξ^α can be changed by arbitrary Lorentz transformations, we can choose a Lorentz boost such that the two coordinate systems are at rest with respect to each other. Then we have

$$\frac{\partial \xi^0}{\partial x^j} = 0, \quad \text{and} \quad \frac{\partial \xi^j}{\partial x^0} = 0 \quad (2.5)$$

for $j = 1, 2, 3$. This leads to

$$g_{00} = \left(\frac{\partial \xi^0}{\partial x^0} \right)^2, \quad g_{jk} = - \sum_{i=1}^3 \frac{\partial \xi^i}{\partial x^j} \frac{\partial \xi^i}{\partial x^k} \quad (2.6)$$

and $g_{0j} = 0$. Now it follows

$$u^0 = \frac{d\xi^0}{d\tau} = \frac{\partial \xi^0}{\partial x^0} \frac{dx^0}{d\tau} = \sqrt{g_{00}} \frac{dx^0}{d\tau}. \quad (2.7)$$

In the following we assume that the non-diagonal elements g_{jk} vanish. Then from the invariant

$$\begin{aligned} u^\alpha \eta_{\alpha\beta} u^\beta &= (u^0)^2 - \vec{u}^2 = 1 = \frac{\partial \xi^\alpha}{\partial x^\mu} \eta_{\alpha\beta} \frac{\partial \xi^\beta}{\partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \\ &= g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = g_{00} \left(\frac{dx^0}{d\tau} \right)^2 + g_{jj} \left(\frac{dx^j}{d\tau} \right)^2 \end{aligned}$$

we find

$$\vec{V}^2 = -\frac{1}{(u^0)^2} g_{jj} \left(\frac{dx^j}{d\tau} \right)^2 = -\sum_{j=1}^3 \frac{g_{jj}}{g_{00}} \frac{(dx^j/d\tau)^2}{(dx^0/d\tau)^2} \quad (2.8)$$

where (2.7) is used.

We want to specialize this for circular motion $r = \text{const}$. Now the geodesic equation (2.3) for $\mu = 1$ reads

$$\frac{d^2 r}{d\tau^2} + \Gamma_{00}^1 \left(\frac{dt}{d\tau} \right)^2 + \Gamma_{11}^1 \left(\frac{dr}{d\tau} \right)^2 + \Gamma_{22}^1 \left(\frac{d\vartheta}{d\tau} \right)^2 + \Gamma_{33}^1 \left(\frac{d\phi}{d\tau} \right)^2 = 0. \quad (2.9)$$

Taking the circular orbit $r = \text{const}$ in the equatorial plane $\vartheta = \pi/2$, then (2.9) gets simplified to

$$\left(\frac{d\phi}{d\tau} \right)^2 = -\frac{\Gamma_{00}^1}{\Gamma_{33}^1} \left(\frac{dt}{d\tau} \right)^2 \quad (2.10)$$

and from (2.8) we finally obtain the important relation

$$\vec{V}^2 = \frac{g_{33}}{g_{00}} \frac{\Gamma_{00}^1}{\Gamma_{33}^1}. \quad (2.11)$$

In the following we consider static spherically symmetric metrics of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^a dt^2 - e^b dr^2 - r^2 e^c (d\vartheta^2 + \sin^2 \vartheta d\phi^2), \quad (2.12)$$

where $a(r), b(r), c(r)$ are functions of r only. For this metric we have [5]

$$\Gamma_{00}^1 = \frac{a'}{2} e^{a-b}$$

$$\Gamma_{33}^1 = -(\frac{r^2}{2}c' + r)e^{c-b} \quad (2.13)$$

here the prime means d/dr . Then (2.11) simply becomes

$$\vec{V}^2 = \frac{a'}{c' + 2/r}. \quad (2.14)$$

3 Solution of the vacuum equation

The metric functions a, b, c appearing in (2.12) must satisfy differential equation which follow from Einstein's equations. In standard general relativity one puts $c = 0$. This is a special choice of gauge which leads to Birkhoff's theorem and the Schwarzschild metric. This works well in the solar system, but obviously not on the galactic scale. The standard way out is to abandon the vacuum equations and assume some hypothetical dark matter. As long as this dark matter is not convincingly recorded one should study the other possibility of retaining $c(r) \neq 0$. Then the vacuum solution is no longer unique. To fix it uniquely we take the expression (2.14) for the circular velocity $V(r)$ as our nonstandard gauge condition. In other words, *every circular velocity $V(r)$ defines a particular gauge*. That means $V(r)$ must be given, so that the theory seems to have less predictive power. What seems to be a weakness is a strength: The asymptotic $V(r)$ cannot be predicted on the basis of the vacuum equations alone, the dynamics of the normal matter, that means the detailed structure of the galaxy, must necessarily be taken into account. Indeed a uniform asymptotic velocity profile for all galaxies seems not to exist. Only with $c \neq 0$ is it possible to carry out our program to express the metric by the observable $V(r)$.

The non-vanishing components of the Ricci tensor for the metric (2.12) are the diagonal elements [5]

$$R_{tt} = \frac{1}{2}e^{a-b}(a'' + \frac{1}{2}a'^2 - \frac{1}{2}a'b' + a'c' + \frac{2}{r}a') \quad (3.1)$$

$$R_{rr} = -\frac{1}{2}(a'' + 2c'') + \frac{b'}{4}(a' + 2c' + \frac{4}{r}) - \frac{a'^2}{4} - \frac{c'^2}{2} - \frac{2}{r}c' \quad (3.2)$$

$$R_{\vartheta\vartheta} = e^{c-b}[-1 - \frac{r^2}{2}c'' - r(2c' + \frac{a' - b'}{2}) - \frac{r^2}{4}c'(a' - b' + 2c')] + 1 \quad (3.3)$$

$$R_{\phi\phi} = \sin^2\vartheta R_{\vartheta\vartheta}, \quad (3.4)$$

the prime always denotes $\partial/\partial r$. Then the Einstein's equations without matter can be reduced to the following three differential equations

$$G_{tt} = e^{a-b} \left[-c'' - \frac{3}{4}c'^2 + \frac{1}{2}b'c' + \frac{1}{r}(b' - 3c') \right] + \frac{1}{r^2}(e^{a-c} - e^{a-b}) = 0 \quad (3.5)$$

$$G_{rr} = \frac{1}{2}a'c' + \frac{1}{r}(a' + c') + \frac{c'^2}{4} + \frac{1}{r^2}(1 - e^{b-c}) = 0 \quad (3.6)$$

$$G_{\vartheta\vartheta} = \frac{r^2}{2}e^{c-b} \left[a'' + c'' - \frac{1}{r}(b' - a' - 2c') + \frac{1}{2}(a'^2 - a'b' + a'c' - b'c' + c'^2) \right] = 0. \quad (3.7)$$

It is not hard to see that there are only two independent field equations. Indeed, using (3.6) b can be expressed by a and c . Eliminating b in (3.5) and (3.7) there results one second order differential equation for a and c :

$$c'' = \frac{a''}{a'} \left(c' + \frac{2}{r} \right) + \frac{4}{r^2} + a'c' + \frac{c'^2}{2} + \frac{2}{r}(a' + c'). \quad (3.8)$$

Introducing the new metric function

$$f(r) = c(r) + 2 \log \frac{r}{r_c} \quad (3.9)$$

where r_c has been included for dimensional reasons, equation (3.8) assumes the simple form

$$\frac{f''}{f'} - \frac{a''}{a'} = a' + \frac{f'}{2}. \quad (3.10)$$

This can immediately be integrated

$$\log \frac{f'}{a'} = a + \frac{f}{2} + \text{const.} \quad (3.11)$$

On the other hand the circular velocity squared (2.14) becomes

$$V^2(r) \equiv u = \frac{a'}{f'}. \quad (3.12)$$

It is the velocity squared $u(r)$ which appears in all equations. Using (3.12) in (3.11) we have

$$a + \frac{c}{2} + \log \frac{r}{r_c} = -\log u. \quad (3.13)$$

Differentiating and eliminating c' by means of (3.9) and (3.12) in the form

$$c' = f' - \frac{2}{r} = \frac{a'}{u} - \frac{2}{r}, \quad (3.14)$$

we find

$$a' = -\frac{2u'}{1+2u}. \quad (3.15)$$

This gives the first diagonal element of the metric

$$g_{tt} = e^a = \frac{K_a}{1+2u} \quad (3.16)$$

where K_a is a constant of integration. Then from (3.13) we get

$$g_{\vartheta\vartheta} = e^c = \left(\frac{1+2u}{u}\right)^2 \frac{K_c}{r^2}, \quad (3.17)$$

where K_c is another integration constant which contains r_c . Finally, $\exp b$ follows from (3.6)

$$g_{rr} = e^b = K_c \left(\frac{u'}{u^2}\right)^2 (1+2u). \quad (3.18)$$

We have succeeded in expressing the metric by the circular velocity squared $u(r)$. If we choose $u = r_s/2(r - r_s)$, we recover the standard gauge $c = 0$ and the Schwarzschild metric. But now also flat rotation curves are possible without dark matter. *As the ether is superfluous in special relativity, so seems to be dark matter in general relativity.* From (3.16-18) we are able to predict other observable quantities which can be computed from the metric, for example lensing data [5]. In this way the theory can be tested. Another test is investigated in the next section.

4 Thin material disk with a dark halo

We study a simple model of a spiral galaxy by assuming that the normal matter is concentrated in the equatorial plane $z = 0$ with a singular density $\sim \delta^1(z)$. For this problem the theory of distribution valued curvature tensor is appropriate which is mainly due to Taub [7]. To be self-contained we give a simple derivation of the relations we need. Let S be a three-dimensional surface in 4-space where the metric tensor $g_{\mu\nu}$ is continuous but has finite jumps in the normal derivatives; the derivatives in the tangential directions are assumed to be continuous. In an admissible coordinate system let S be described by the equation

$$\varphi(x) = 0 \quad (4.1)$$

and have the normal vector

$$n_\mu = \frac{\partial \varphi}{\partial x^\mu}. \quad (4.2)$$

Then the finite discontinuities in the first partial derivatives of $g_{\mu\nu}$ are given by

$$[g_{\mu\nu,\sigma}] \equiv \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \Big|_+ - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \Big|_- = n_\sigma b_{\mu\nu}, \quad (4.3)$$

where $+$ and $-$ mean the limiting values from both sides of S . This follows from the decomposition of the gradient into normal and tangential components. The corresponding jumps in the Christoffel symbols then are

$$2[\Gamma_{\beta\gamma}^\alpha] = n_\beta b_\gamma^\alpha + n_\gamma b_\beta^\alpha - n^\alpha b_{\beta\gamma}. \quad (4.4)$$

The Ricci tensor

$$R_{\mu\nu} = \partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\alpha}^\alpha + \Gamma_{\alpha\beta}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\nu\beta}^\alpha \Gamma_{\alpha\mu}^\beta \quad (4.5)$$

contains derivatives of Γ , consequently the finite jumps lead to singular contributions proportional to the delta distribution δ_S with support on S according to

$$\partial_\beta \Gamma_{\mu\nu}^\alpha|_{\text{sing}} = [\Gamma_{\mu\nu}^\alpha] n_\beta \delta_S. \quad (4.6)$$

Then it follows from (4.4) that

$$R_{\mu\nu}|_{\text{sing}} = \frac{1}{2}(-n^\alpha n_\alpha b_{\mu\nu} + n^\alpha \tilde{b}_{\mu\alpha} n_\nu + n^\alpha \tilde{b}_{\alpha\nu} n_\mu) \delta_S, \quad (4.7)$$

with

$$\tilde{b}_\beta^\alpha = b_\beta^\alpha - \frac{1}{2} b \delta_\beta^\alpha, \quad b = g^{\mu\nu} b_{\mu\nu}. \quad (4.8)$$

This is in agreement with eq.(2.14) of Taub, note that his convention for the Ricci tensor is the negative of our (4.5).

In the Einstein's equations these singular distribution must be compensated by a distribution valued energy-momentum tensor

$$(R_{\mu\nu} - \frac{1}{2} R)|_{\text{sing}} = \kappa t_{\mu\nu} \delta_S, \quad (4.9)$$

where

$$R = g^{\alpha\beta} R_{\alpha\beta}, \quad \kappa = \frac{8\pi G}{c^2}. \quad (4.10)$$

If the jumps $b_{\mu\nu}$ of the normal derivatives of $g_{\mu\nu}$ are known, $t_{\mu\nu}$ can be calculated from (4.7) and (4.9):

$$-2\kappa t_{\mu\nu} = n^2 \left((g_\mu^\sigma - \frac{n^\sigma n_\mu}{n^2}) (g_\nu^\tau - \frac{n^\tau n_\nu}{n^2}) - \right.$$

$$-(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2})(g^{\sigma\tau} - \frac{n^\sigma n^\tau}{n^2})b_{\sigma\tau}, \quad (4.11)$$

where $n^2 = n^\alpha n_\alpha$. This agrees with eq.(6-2) of Taub. The singular contribution (4.11) must be added to the regular energy-momentum tensor which renders the field equations fulfilled outside of the surface S .

Now we come to our simple galaxy model where the normal matter is concentrated in the plane $z = 0$ which is our singular surface S . Outside this plane we have vacuum with a dark halo as it is described by the nonstandard spherically symmetric solution (3.16-18). To have a simple representation of the plane $z = 0$ and the corresponding delta-measure we go over to cylindrical coordinates (t, R, z, ϕ)

$$r^2 = R^2 + z^2, \quad z = r \cos \vartheta, \quad \sin \vartheta = \frac{R}{r}. \quad (4.12)$$

Then the metric (2.12) assumes the following non-diagonal form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

with

$$\begin{aligned} g_{00} &= e^a, & g_{11} &= -\frac{R^2}{r^2}e^b - \frac{z^2}{r^2}e^c, & g_{22} &= -\frac{R^2}{r^2}e^c - \frac{z^2}{r^2}e^b \\ g_{12} &= g_{21} = -2\frac{rz}{r^2}(e^b - e^c), & g_{33} &= -R^2e^c. \end{aligned} \quad (4.13)$$

For simplicity we still write r , but our admissible coordinates are $x^1 = R, x^2 = z$. We also need the inverse

$$\begin{aligned} g^{00} &= e^{-a}, & g^{11} &= \frac{g_{22}}{D}, & g^{22} &= \frac{g_{11}}{D} \\ g^{12} &= -\frac{g_{12}}{D} = g^{21}, & g^{33} &= \frac{1}{g_{33}}, \end{aligned} \quad (4.14)$$

where the determinant D is equal to

$$D = g_{11}g_{22} - (g_{12})^2 = e^{b+c} - 3\frac{R^2 z^2}{r^4}(e^b - e^c)^2. \quad (4.15)$$

To construct the metric with the material disk we apply the widely used displace, cut, and reflect method which goes back to Kuzmin [7] and since then has been used by many authors. Following the procedure of Voigt and Letelier [8] we take the metric (3.7) in the half space $z > d > 0$, displace it

to $z = 0$ and reflect it for $z < 0$. This produces the finite jumps in the z -derivatives of $g_{\mu\nu}$. The whole procedure is equivalent to the transformation $z \rightarrow |z| + d$. The normal vector is $n_\mu = (0, 0, 1, 0) = \delta_\mu^2$ and

$$n^\nu = g^{\nu\mu} n_\mu = g^{\nu 2}, \quad n^\nu n_\nu = g^{22}.$$

The jumps (4.3) in the normal derivatives on $z = 0$ which we need are equal to

$$b_{11} = [g_{11,2}] = g'_{11} \frac{2d}{r} - \frac{4d}{r^2} e^c \quad (4.16)$$

$$b_{33} = [g_{33,2}] = g'_{33} \frac{2d}{r},$$

where the prime always means $\partial/\partial r$ keeping z and R constant. Now from (4.11) we find the energy density

$$t_0^0 = \frac{1}{2\kappa} \left(D b_{11} + \frac{g_{11}}{D g_{33}} b_{33} \right) \quad (4.17)$$

with $D = g_{11} g_{22} - g_{12}^2$. Using

$$b_{11} = \frac{2d}{r} \left(g'_{11} - \frac{2}{r} e^c \right), \quad b_{33} = \frac{2d}{r} g'_{33} c',$$

we finally obtain

$$t_0^0 = -\frac{d}{\kappa r} \left(e^{b+c} \frac{R^6}{r^4} \partial_r \left(\frac{e^b}{r^2} \right) + \frac{2R^4}{r^5} e^{b+2c} - \frac{r^2}{R^2} \partial_r e^{-c} \right). \quad (4.18)$$

Here we have to put $z = 0$ everywhere which gives $r^2 = R^2 + d^2$.

Now we must specify the circular velocity squared $u(r)$ in order to fix the metric. We are particularly interested in the case of an asymptotically flat circular velocity which in the usual terminology corresponds to a dark halo. Therefore we assume $u(r)$ of the form

$$u(r) = u_{\text{flat}} + \frac{u_1}{r} + O(r^{-2}) \quad (4.19)$$

for large r . Then it follows from (4.16-18)

$$\begin{aligned} e^a &= K_a + O(r^{-1}), \quad e^b = \frac{L_b}{r^4} + O(r^{-5}) \\ e^c &= \frac{L_c}{r^2} + O(r^{-3}) \end{aligned} \quad (4.20)$$

where by (3.17)

$$L_c \sim u_{\text{flat}}^{-2} = V_{\text{flat}}^{-4}. \quad (4.21)$$

Using this in (4.18) the leading order comes from the last term

$$t_0^0 = \frac{2d}{\kappa L_c} \frac{r^2}{R^2} (1 + O(R^{-1})). \quad (4.22)$$

This is proportional to the density of normal matter because we consider a static energy-momentum tensor. Taking (4.21) into account we find that

$$t_0^0 \sim u_{\text{flat}}^2 \sim V_{\text{flat}}^4(R) \quad (4.23)$$

for large R . This is in accordance with the baryonic Tully-Fisher relation for galaxies [2] [3], which states that the total baryonic mass M is proportional to V_{flat}^4 . In fact, one can show [10] that the contribution of the inner part $R < R_1$ of the disk can be made arbitrarily small compared to the outer part between $R_1 < R < R_2$, say. We emphasize that M is obtained from t_0^0 by integrating with the Euclidean surface measure $R dR d\phi$, because this is what astronomers are doing when they determine M from luminosity measurements. Our theory gives a very natural explanation of the Tully-Fisher relation which, otherwise, theoretically and observationally is somewhat mysterious.

The radial pressure t_r^r vanishes because G_{rr} (3.6) does not contain a second derivative. Therefore our model must be interpreted as a dust disk with purely azimuthal stresses. This is not very realistic and it remains to be investigated whether the Tully-Fisher relation is a generic property for more physical galaxy models.

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